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Chebyshev Polynomials Are Not Always Optimal*

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We are concerned with the problem of finding the polynomial with minimal uniform norm on $\mathscr{E}$ among all polynomials of degree at most n and normalized to be 1 at c. Here, $\mathscr{E}$ is a given ellipse with both foci on the real axis and c is a given real point not contained in $\mathscr{E}$. Problems of this type arise in certain iterative matrix computations, and, in this context, it is generally believed and widely referenced that suitably normalized Chebyshev polynomials are optimal for such constrained approximation problems. In this work, we show that this is not true in general. Moreover, we derive sufficient conditions which guarantee that Chebyshev polynomials are optimal. Also, some numerical examples are presented.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let Π_n be the set of all complex polynomials of degree at most n. For $r > 1$, we denote by

$$\mathscr{E}_r := \left\{ z \in \mathbb{C} \;\middle|\; |z-1| + |z+1| \leqslant r + \frac{1}{r} \right\}$$

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the ellipse with foci at ± 1 and semi-axes

$$a_r := \frac{1}{2}\left(r + \frac{1}{r}\right), \qquad b_r := \frac{1}{2}\left(r - \frac{1}{r}\right).$$

In this work, we study the constrained Chebyshev approximation problem

$$\min_{p \in \Pi_n : p(c) = 1} \max_{z \in \mathscr{E}_r} |p(z)|, \tag{1}$$

where $n \in \mathbb{N}$, $r > 1$, and $c \in \mathbb{R} \backslash \mathscr{E}_r$. Standard results from approximation theory (see, e.g., [9]) show that there always exists a unique optimal polynomial, denoted by $p_n(z; r, c)$ in the sequel, for (1) and, moreover, that p_n is a real polynomial. In 1963, Clayton [3] proved that $p_n(z; r, c)$ is just the polynomial

$$t_n(z; c) := \frac{T_n(z)}{T_n(c)}, \tag{2}$$

where

$$T_n(z) = \frac{1}{2}\left(v^n + \frac{1}{v^n}\right), \qquad z = \frac{1}{2}\left(v + \frac{1}{v}\right) \tag{3}$$

denotes the nth Chebyshev polynomial. The approximation problem (1) arises in certain iterative matrix computations (see, e.g., [2, 5]). In this context, Clayton's result is widely referenced in the literature (e.g., [2, 5, 8, 12, 13]) and is even used to derive new results on constrained approximation problems [1]. Surprisingly, nobody seems to have checked Clayton's proof.

In this note, we show that the normalized Chebyshev polynomials (2) are *not* always optimal for (1), and hence Clayton's result is not true in general. More precisely, we have the following

THEOREM 1. (a) *Let $r > 1$ and $c > a_r$ or $c < -a_r$. Then, for $n = 1, 2, 3, 4$, $t_n(z; c)$ is the unique optimal polynomial for* (1).

(b) *For any integer $n \geqslant 5$ there exists a real number $r^* = r^*(n) > 1$ such that $t_n(z; c)$ is* not *optimal for* (1) *for all $r > r^*$ and all $c \in \mathbb{R}$ with $a_r < |c| \leqslant a_r + 1/a_r^2$.*

However, $t_n \equiv p_n$ in most cases, and t_n ceases to be optimal only for normalization points c which are very close to the ellipse. We show that the following conditions on c are sufficient to guarantee the optimality of t_n.

THEOREM 2. *Let $n \geqslant 5$ be an integer, $r > 1$, and $c \in \mathbb{R}$. Then, $t_n(z; c)$ is the unique optimal polynomial for* (1) *if*

(a) $|c| \geqslant \frac{1}{2}(r^{\sqrt{2}} + r^{-\sqrt{2}})$ *or*

(b) $|c| \geqslant (1/2a_r)(2a_r^2 - 1 + \sqrt{2a_r^4 - a_r^2 + 1})$.

Remark 1. In general, the conditions (a) and (b) do not imply each other. In particular, (a) (resp. (b)) is less stringent for small r (resp. large r). Also, note that (b) is satisfied if $|c| \geqslant (1 + \sqrt{2}/2)a_r$.

The paper is organized as follows. In Section 2, we state a necessary and sufficient criterion for t_n to be optimal for (1). Also some auxiliary results are collected which are used in Section 3 and 4 to prove Theorem 1 and 2, respectively. Finally, in Section 5, we present some numerical examples.

2. PRELIMINARIES

In the sequel, let always $r > 1$ and $n \in \mathbb{N}$. Since $p_n(z; r, -c) \equiv p_n(-z; r, c)$ it is sufficient to consider positive c only; so for the rest of the paper, we assume that $c > a_r$.

First, we determine the extremal points z_l of t_n defined by

$$|t_n(z_l; c)| = \max_{z \in \mathscr{E}_r} |t_n(z; c)|, \qquad z_l \in \mathscr{E}_r.$$

With (3), one easily verifies that there are $2n$ such points given by

$$z_l := a_r \cos \varphi_l + ib_r \sin \varphi_l, \qquad \varphi_l := l\pi/n, \qquad l = 1, ..., 2n.$$

Moreover, note that $t_n(z_l; c) = (-1)^l\, T_n(a_r)/T_n(c)$. Using Rivlin and Shapiro's characterization [10] of the optimal solution of general linear Chebyshev approximation problems, we deduce that $t_n \equiv p_n$ iff there exist nonnegative real numbers σ_l, $l = 1, ..., 2n$ (not all zero), such that

$$\sum_{l=1}^{2n} \sigma_l (-1)^l q(z_l) = 0 \qquad \text{for all} \quad q \in \Pi_n \quad \text{with} \quad q(c) = 0. \tag{4}$$

By solving this linear system explicitly, one arrives at the following

LEMMA 1. *The polynomial t_n in* (2) *is optimal for* (1) *iff $\sigma_l \geqslant 0$ for $l = 1, ..., 2n$, where*

$$\sigma_l := (-1)^l \left(\frac{1}{2} \left(1 + (-1)^l \frac{T_n(c)}{T_n(a_r)} \right) + \sum_{k=1}^{n-1} \frac{T_k(c)}{T_k(a_r)} \cos(k\varphi_l) \right). \tag{5}$$

Proof. The result is a special case of Theorem 3 in [4], where we investigated the approximation problem (1) in the more general setting of complex c. On the other hand, by using the polynomials $q(z) = T_k(z) - T_k(c)$, $k = 1, ..., n$, as a basis in (4), it is also straightforward to verify directly that the σ_l given by (5) satisfy (4) and that these are up to a constant factor the only solutions of (4). ∎

Remark 2. Clearly $\sigma_{2n} > 0$ and, moreover, $\sigma_l = \sigma_{2n-l}$. Hence, t_n is optimal iff $\sigma_l \geq 0$ for $l = 1, ..., n$.

The following result due to Rogosinski and Szegő [11] is used in the next section to establish a sufficient condition for the positivity of the σ_l.

LEMMA 2. *Let $\lambda_0, \lambda_1, ..., \lambda_n$ be real numbers which satisfy $\lambda_n \geq 0$, $\lambda_{n-1} - 2\lambda_n \geq 0$, and $\lambda_{k-1} - 2\lambda_k + \lambda_{k+1} \geq 0$ for $k = 1, 2, ..., n-1$. Then*

$$s(\varphi) := \frac{\lambda_0}{2} + \sum_{k=1}^{n} \lambda_k \cos(k\varphi) \geq 0 \qquad \text{for all} \quad \varphi \in \mathbb{R}. \tag{6}$$

We close this section with the following technical lemma. The proof is straightforward and omitted here.

LEMMA 3. (a) *Let $k \in \mathbb{N}$. Then*

$$\sum_{j=1}^{k} \cos^2 \frac{(j-1/2)\pi}{k} = \begin{cases} 0 & \text{if} \quad k = 1 \\ k/2 & \text{if} \quad k \geq 2. \end{cases}$$

(b) *Let $2 \leq l \leq n$ be an even integer and $\varphi_l = l\pi/n$. Then*

$$\sum_{k=0}^{n-1} \cos(k\varphi_l) = 0 \tag{7}$$

and

$$\sum_{k=1}^{n-1} k \cos(k\varphi_l) = -n/2. \tag{8}$$

3. PROOF OF THEOREM 1

Let $r > 1$ be fixed and set $a := a_r$. Then, for each l, (5) defines a polynomial $\sigma_l(c) = \sigma_l$ in c of degree n. Therefore,

$$\sigma_l(c) = \sigma_l(a) + (c-a)\left(\sigma_l'(a) + \sum_{j=2}^{n} \frac{\sigma_l^{(j)}(a)}{j!}(c-a)^{j-1}\right). \tag{9}$$

First, we prove part (b) of Theorem 1. Let $n \geqslant 5$ and $2 \leqslant l \leqslant n$ be an even integer. With (5) and (7), it follows that

$$\sigma_l(a) = (-1)^l \left(\frac{1}{2}(1+(-1)^l) + \sum_{k=1}^{n-1} \cos(k\varphi_l) \right) = 0. \tag{10}$$

Furthermore, we derive from (5) that

$$\sigma_l'(a) = \frac{1}{2} \frac{T_n'(a)}{T_n(a)} + \sum_{k=1}^{n-1} \frac{T_k'(a)}{T_k(a)} \cos(k\varphi_l). \tag{11}$$

Let $\xi_j^{(k)} = \cos((2j-1)\pi/(2k))$, $j = 1, ..., k$, denote the zeros of T_k. Then,

$$\begin{aligned}
\frac{T_k'(a)}{T_k(a)} &= \sum_{j=1}^{k} \frac{1}{a-\xi_j^{(k)}} = \sum_{m=0}^{\infty} \frac{1}{a^{m+1}} \sum_{j=1}^{k} (\xi_j^{(k)})^m \\
&= \sum_{m=0}^{\infty} \frac{1}{a^{2m+1}} \sum_{j=1}^{k} (\xi_j^{(k)})^{2m} \\
&= k/a + \begin{cases} 0 & \text{if} \quad k=1 \\ k/(2a^3) + O(1/a^5) & \text{if} \quad k \geqslant 2. \end{cases}
\end{aligned} \tag{12}$$

Here, we used the fact that T_k'/T_k is an odd function and part (a) of Lemma 3. With (8), (11), and (12), it follows that

$$\sigma_l'(a) = -\frac{1}{2} \cos\left(\frac{l\pi}{n}\right) \frac{1}{a^3} + O\left(\frac{1}{a^5}\right). \tag{13}$$

Combining (9), (10), and (13) yields

$$\sigma_l(c) = (c-a)\left(-\frac{1}{2}\cos\left(\frac{l\pi}{n}\right)\frac{1}{a^3} + O\left(\frac{1}{a^5}\right) + \sum_{j=2}^{n} \frac{\sigma_l^{(j)}(a)}{j!}(c-a)^{j-1} \right)$$

and, finally, since, given (5) and $T_k^{(j)}(a)/T_k(a) = O(1/a^j)$, for $j \geqslant 2$ we have $\sigma_l^{(j)}(a) = O(1/a^2)$,

$$\sigma_l(c) = \frac{c-a}{a^3}\left(-\frac{1}{2}\cos\left(\frac{l\pi}{n}\right) + O\left(\frac{1}{a^2}\right) + O(a(c-a)) \right).$$

Thus, $\sigma_l(c) < 0$ and, therefore, (2) is not the optimal polynomial for (1), if $c - a \leqslant 1/a^2$, a is sufficiently large, and $\cos(l\pi/n) > 0$, i.e., $l < n/2$. Note that even l with $2 \leqslant l < n/2$ exist, since $n \geqslant 5$. This concludes the proof of part (b) of Theorem 1.

We now turn to the proof of part (a) of Theorem 1. Let $r > 1$ and

$c > a = a_r$ be fixed. Moreover, set $A_k := T_k(c)$ and $a_k := T_k(a)$. Then, in view of Lemma 1 and Remark 2, one needs to check the positivity of

$$\sigma_l^{(n)} = (-1)^l \left(\frac{1}{2} \left(1 + (-1)^l \frac{A_n}{a_n} \right) + \sum_{k=1}^{n-1} \frac{A_k}{a_k} \cos\left(\frac{kl\pi}{n} \right) \right), \qquad l = 1, ..., n, \qquad (14)$$

for the four cases $n = 1, 2, 3, 4$. For $n = 1, 2$ this is clearly true, since

$$\sigma_1^{(1)} = \frac{1}{2} \left(\frac{A_1}{a_1} - 1 \right) > 0, \qquad \sigma_1^{(2)} = \frac{1}{2} \left(\frac{A_2}{a_2} - 1 \right) > 0,$$

and

$$\sigma_2^{(2)} = \frac{1}{2} \left(\frac{A_2}{a_2} - 2\frac{A_1}{a_1} + 1 \right) = \frac{(c-a)(ac - a^2 + 1)}{a(2a^2 - 1)} > 0.$$

Next, consider $n = 3$. It is easily verified that $A_3/a_3 > A_1/a_1$, and hence

$$\sigma_1^{(3)} = \frac{1}{2} \left(\frac{A_3}{a_3} - \frac{A_1}{a_1} \right) + \frac{1}{2} \left(\frac{A_2}{a_2} - 1 \right) > 0.$$

By using that $T_2(c)\, T_2(a) + ca$ is a monotonously increasing function in c for $c \geqslant a \geqslant 1$, we deduce that

$$\begin{aligned} \sigma_2^{(3)} &= \frac{1}{2} \left(\frac{A_3}{a_3} - \frac{A_2}{a_2} - \frac{A_1}{a_1} + 1 \right) \\ &= \frac{(c-a)}{2a} \left(\frac{2T_2(c)\, T_2(a) + 2ca + 1}{(4a^2 - 3)(2a^2 - 1)} - 1 \right) \\ &\geqslant \frac{2a(c-a)}{(4a^2 - 3)(2a^2 - 1)} > 0. \end{aligned}$$

Similarly, one obtains

$$\begin{aligned} \sigma_3^{(3)} &= \frac{1}{2} \frac{A_3}{a_3} - \frac{A_2}{a_2} + \frac{A_1}{a_1} - \frac{1}{2} \\ &= \frac{(c-a)}{a} \left(\frac{4(c^2 + 2ca + a^2) - 3}{2(4a^2 - 3)} - \frac{2ca + 1}{2a^2 - 1} \right) \\ &\geqslant \frac{(c-a)(16a^4 - 18a^2 + 9)}{2a(4a^2 - 3)(2a^2 - 1)} > 0. \end{aligned}$$

Finally, we turn to the case $n = 4$. Analogously to the case $n = 3$, $l = 1$,

$$\sigma_1^{(4)} = \frac{1}{2} \left(\frac{A_4}{a_4} - 1 \right) + \frac{\sqrt{2}}{2} \left(\frac{A_3}{a_3} - \frac{A_1}{a_1} \right) > 0.$$

For $l=2$, we have

$$\sigma_2^{(4)}=\frac{1}{2}\left(\frac{A_4}{a_4}-2\frac{A_2}{a_2}+1\right)=\frac{(A_2-a_2)(A_2a_2-a_2^2+1)}{a_2(2a_2^2-1)}>0.$$

The positivity of $\sigma_3^{(4)}$ follows from

$$\begin{aligned}\frac{\sigma_3^{(4)}}{2(c^2-a^2)}&=\frac{1}{4(c^2-a^2)}\left(\frac{A_4}{a_4}-1-\sqrt{2}\left(\frac{A_3}{a_3}-\frac{A_1}{a_1}\right)\right)\\&=\frac{2(c^2+a^2-1)}{8a^4-8a^2+1}-\frac{\sqrt{2}c}{a(4a^2-3)} \qquad (15)\\&\geqslant\frac{8(2-\sqrt{2})a^4+4(2\sqrt{2}-5)a^2+6-\sqrt{2}}{(8a^4-8a^2+1)(4a^2-3)}>0. \qquad (16)\end{aligned}$$

Here we have used that (15) is a monotonously increasing function in c for $c\geqslant 1$ and that the numerator in (16) has no real zero. Similarly, by a routine, but lengthy, computation, one verifies that

$$\begin{aligned}\frac{a_2a_3a_4}{2(c-a)}\sigma_4^{(4)}&=\frac{a_2a_3a_4}{2(c-a)}\left(\frac{1}{2}\frac{A_4}{a_4}-\frac{A_3}{a_3}+\frac{A_2}{a_2}-\frac{A_1}{a_1}+\frac{1}{2}\right)\\&=(2c^2-1)((c-a)a_3+a_2)a_2\\&\quad+((c(4a^2-1)-a_3)(a_2-1)a-a_2)(a_2-1)\\&\geqslant a_2(4a^4-6a^2+3)+2a^2(a_2-1)^2>0.\end{aligned}$$

This concludes the proof of part (a) of Theorem 1.

4. Proof of Theorem 2

Let $r>1$ and $c>a:=a_r$ be fixed. Note that a and c have the representations

$$a=\frac{1}{2}\left(r+\frac{1}{r}\right),\qquad c=\frac{1}{2}\left(R+\frac{1}{R}\right),\qquad R>r. \qquad (17)$$

With (3) and (17), one obtains

$$\frac{T_k(c)}{T_k(a)}=\frac{R^k+1/R^k}{r^k+1/r^k}=f(\varphi_k), \qquad (18)$$

where we set

$$f(\varphi):=\frac{\cosh((\log R)\,n\varphi/\pi)}{\cosh((\log r)\,n\varphi/\pi)},\qquad \varphi_k:=\frac{k\pi}{n}.$$

Since f is continuous, bounded, and even, it can be expanded into the Fourier series

$$f(\varphi)=\frac{1}{2}\alpha_0+\sum_{j=1}^{\infty}\alpha_j\cos(j\varphi), \qquad -\pi\leqslant\varphi\leqslant\pi.$$

By rewriting the expression (5) for σ_l in terms of (18) and, subsequently, using the discrete orthogonality relations of $\cos(l\varphi_k)$, $k, l=0, ..., n$ (see, e.g., [7, p. 472]), we get

$$\begin{aligned}\sigma_l&=(-1)^l\left(\frac{1}{2}(f(0)+(-1)^l f(\pi))+\sum_{k=1}^{n-1} f(\varphi_k)\cos(l\varphi_k)\right)\\ &=\begin{cases}\dfrac{n}{2}(-1)^l\left(\alpha_l+\displaystyle\sum_{m=1}^{\infty}(\alpha_{2mn-l}+\alpha_{2mn+l})\right) & \text{for} \quad l=1, ..., n-1\\ n(-1)^l\left(\alpha_n+\displaystyle\sum_{m=1}^{\infty}\alpha_{2(m+1)n}\right) & \text{for} \quad l=n.\end{cases}\end{aligned}$$

It follows that all $\sigma_l\geqslant 0$ and, in view of Lemma 1, that the normalized Chebyshev polynomials (2) are optimal for (1), if the Fourier coefficients α_j of f satisfy

$$\alpha_j=(-1)^j|\alpha_j|, \qquad j=1, 2, ... \tag{19}$$

It is well known (see, e.g., [6, Theorem 35]) that (19) holds true if f is a convex function. Hence, in order to prove that the condition (a) in Theorem 2 guarantees the optimality of the polynomial (2) for (1), it only remains to show that (a) implies the convexity of f. Since f is even, we only need to consider $\varphi\geqslant 0$. Moreover, set $x:=(\log r)\,n\varphi/\pi$ and $\gamma:=\log R/\log r>1$. Then, using standard calculus, we obtain

$$\begin{aligned}&\frac{\cosh(x)}{\cosh(\gamma x)}\left(\frac{\pi}{n\log r}\right)^2 f''(\varphi)\\ &\quad=\gamma^2-1-2\gamma\tanh(x)\tanh(\gamma x)+2\tanh^2(x)\\ &\quad\geqslant\gamma^2-1-2\gamma\tanh(x)+2\tanh^2(x)\\ &\quad\geqslant\gamma^2-1+2\min_{0\leqslant y\leqslant 1} y(y-\gamma)\\ &\quad=\begin{cases}(1-\gamma)^2 & \text{if} \quad \gamma>2\\ \gamma^2/2-1 & \text{if} \quad \gamma\leqslant 2.\end{cases}\end{aligned} \tag{20}$$

Therefore, (20) is nonnegative, and thus f convex, if $\gamma\geqslant\sqrt{2}$. This last condition is easily seen to be equivalent to the condition (a) in Theorem 2.

Remark 3. The main idea of the proof, namely, to verify the positivity of the σ_l via the convexity of f, is due to Clayton [3]. However, in [3], it is claimed that f is convex in all cases $R > r > 1$. Unfortunately, this is not true in general.

Now, assume that condition (b) of Theorem 2 is fulfilled. Again, we use the notations $A_k = T_k(c)$ and $a_k = T_k(a)$. Note that, by the three-term recurrence formula of the Chebyshev polynomials,

$$A_{k+1} = 2cA_k - A_{k-1}, \qquad k = 1, 2, ... \tag{21}$$

Next, set

$$\lambda_0 = \frac{A_n}{a_n}, \quad \lambda_n = \frac{1}{2}, \quad \text{and,} \quad \text{for} \quad k = 1, 2, ..., n-1, \quad \lambda_k = \frac{A_{n-k}}{a_{n-k}}, \tag{22}$$

and let $s(\varphi)$ be the trigonometric polynomial defined by (6). With (5) and (6), one readily verifies that $\sigma_l = s(l\pi/n)$, and, in view of Lemmas 1 and 2, we conclude that the polynomial (2) is indeed optimal for (1) if the numbers (22) satisfy

$$\begin{gathered} \lambda_n \geqslant 0, \quad \lambda_{n-1} - 2\lambda_n \geqslant 0, \quad \text{and,} \\ \text{for} \quad k = 1, ..., n-1, \quad \lambda_{k-1} - 2\lambda_k + \lambda_{k+1} \geqslant 0. \end{gathered} \tag{23}$$

The first condition in (23) is trivially true, and the second one follows from $A_1 > a_1$. Using (22), the remaining inequalities in (23) can be rewritten in the form

$$\frac{A_2}{a_2} - 2\frac{A_1}{a_1} + \frac{1}{2} \geqslant 0 \tag{24}$$

and

$$\frac{A_{j+1}}{a_{j+1}} - 2\frac{A_j}{a_j} + \frac{A_{j-1}}{a_{j-1}} \geqslant 0, \qquad \text{for} \quad j = 2, ..., n-1. \tag{25}$$

A simple calculation shows that (24) is equivalent to

$$c \geqslant c^* := \frac{a_2 + \sqrt{a^2 a_2 + 1}}{2a} \qquad \left(= \frac{2a_r^2 - 1 + \sqrt{2a_r^4 - a_r^2 + 1}}{2a_r} \right), \tag{26}$$

which is just condition (b). For the proof of Theorem 2, it only remains to show that (26) also implies (25). Let $j \geqslant 2$. First, by using (21), we deduce that

$$\begin{aligned}
&\frac{A_{j+1}}{a_{j+1}} - 2\frac{A_j}{a_j} + \frac{A_{j-1}}{a_{j-1}} \\
&\quad = A_j\left(2\left(\frac{c}{a_{j+1}} - \frac{1}{a_j}\right) + \frac{1}{2c}\left(\frac{1}{a_{j-1}} - \frac{1}{a_{j+1}}\right)\right) + \frac{A_{j-2}}{2c}\left(\frac{1}{a_{j-1}} - \frac{1}{a_{j+1}}\right) \\
&\quad \geqslant \frac{A_j}{2ca_{j+1}a_ja_{j-1}}\left(4c^2a_ja_{j-1} - 4ca_{j+1}a_{j-1} + a_j(a_{j+1} - a_{j-1})\right).
\end{aligned} \tag{27}$$

Next, set

$$Q_j(c) := 4c^2a_ja_{j-1} - 4ca_{j+1}a_{j-1} + a_j(a_{j+1} - a_{j-1})$$

and note that Q_j attains its minimum at $a_{j+1}/(2a_j) < c^*$. Hence, in view of (27), (25) holds true, if $Q_j(c^*) \geqslant 0$ is fulfilled. This is indeed the case, and we show by induction that

$$Q_j(c^*) \geqslant Q_2(c^*) \geqslant 0, \qquad j = 2, 3, \ldots. \tag{28}$$

For $j = 2$, this follows with

$$\begin{aligned}
Q_2(c^*) &= 4(c^*)^2\, a_2a - 4c^*a_3a + a_2(a_3 - a) \\
&= a^{-1}\left(a_2(2a^4 - 3a^2 + 2) - (a_2 - 1)\sqrt{a^2a_2 + 1}\right) \geqslant 0,
\end{aligned}$$

since $\sqrt{2}a_2 \geqslant \sqrt{a^2a_2 + 1}$ and $2a^4 - 3a^2 + 2 \geqslant \sqrt{2}(a_2 - 1)$ for $a \geqslant 1$. Finally, if (28) holds true for j, a routine, but lengthy, calculation shows that

$$\begin{aligned}
&Q_{j+1}(c^*) - Q_j(c^*) \\
&\quad = (a_2 - 1)\left(-4(c^*)^2\, a + 2c^*\frac{a_{j+2}}{a_j} + a\right) + \left(\frac{a_{j+2}}{a_j} - 1\right) Q_j(c^*) \\
&\quad \geqslant (a_2 - 1)\left(-4(c^*)^2\, a + 2c^*\frac{a_4}{a_2} + a\right) + \left(\frac{a_4}{a_2} - 1\right) Q_2(c^*) \\
&\quad = (a_2 - 1)(2(Q_2(c^*) - c^*) + a_3) \geqslant 0
\end{aligned}$$

(note that $a_{j+2}/a_j \geqslant a_4/a_2$). Therefore, (28) is also satisfied for $j+1$, and this completes the proof of Theorem 2.

5. Some Numerical Examples

In order to illustrate the range of parameters for which the normalized Chebyshev polynomials (2) are not optimal for the approximation problem

TABLE I

The numerically computed values of $r^* := r^*(n)$ and the corresponding semi-axes a_{r^*} and b_{r^*} of the ellipse $\mathscr{E}_{r^*}$ are listed for $n = 5, 6, \ldots, 20$

n	r^*	a_{r^*}	b_{r^*}	n	r^*	a_{r^*}	b_{r^*}
5	2.6492	1.5133	1.1359	13	1.3402	1.0432	0.2970
6	2.0588	1.2723	0.7865	14	1.3111	1.0369	0.2742
7	1.8006	1.1780	0.6226	15	1.2867	1.0319	0.2547
8	1.6490	1.1277	0.5213	16	1.2658	1.0279	0.2379
9	1.5476	1.0969	0.4508	17	1.2478	1.0246	0.2232
10	1.4745	1.0764	0.3982	18	1.2321	1.0219	0.2103
11	1.4191	1.0619	0.3574	19	1.2183	1.0196	0.1988
12	1.3755	1.0512	0.3242	20	1.2061	1.0176	0.1885

(1), we present a few numerical examples. Let $r^* = r^*(n)$ denote the smallest $r > 1$ such that for all $r > r^*$ there exists a real number $c(r, n) > a_r$ such that for all $a_r < c < c(r, n)$ the polynomial (2) is not best possible in (1). For later use, let us denote by $c^*(r, n)$ the maximal $c(r, n)$ with this property. Recall that in view of Theorems 1 and 2, $1 < r^*(n) < \infty$ exists for all integers $n \geqslant 5$. In Table I, the numerically computed values of $r^*(n)$ and the corresponding semi-axes of $\mathscr{E}_{r^*}$ are listed for $5 \leqslant n \leqslant 20$. Note that $r^*(n)$ tends to 1 as n increases.

The case where the normalized Chebyshev polynomials (2) are not optimal for (1) occurs only for c close to the ellipse. In Fig. 1, for the cases

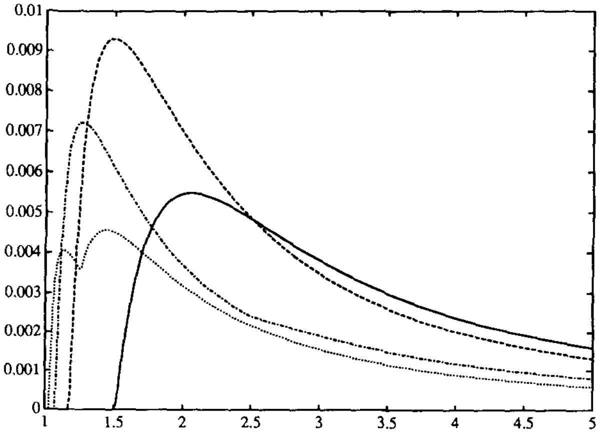

FIG. 1. The functions $f_n(a_r) := (c^*(r, n) - a_r)/a_r$ are plotted in the range $1 \leqslant a_r \leqslant 5$ for the cases $n = 5$ (solid line), $n = 7$ (dashed line), $n = 10$ (dash–dotted line), and $n = 15$ (dotted line).

$n = 5$ (solid line), $n = 7$ (dashed line), $n = 10$ (dash–dotted line), and $n = 15$ (dotted line), the curves

$$\frac{c^*(r, n) - a_r}{a_r}$$

are plotted as functions of a_r.

For some cases for which (2) is not optimal for (1), we computed the best polynomials numerically. We were not able to detect any analytic representation of these polynomials.

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